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# Some topics in $L^p$ -theory for second-order elliptic operators with unbounded coefficients

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## 1. Introduction

In this paper we deal with the second-order elliptic operators of the form

$$(1.1) \quad Au(x) := -\operatorname{div}(a(x)\nabla u(x)) + F(x) \cdot \nabla u(x) + V(x)u(x), \quad x \in \mathbb{R}^N,$$

where  $N \in \mathbb{N}$  and the coefficients  $(a, F, V)$  satisfy the following condition:

(A1)  ${}^t a = a \in C^1(\mathbb{R}^N; \mathbb{R}^{N \times N})$ ,  $F \in C^1(\mathbb{R}^N; \mathbb{R}^N)$ ,  $V \in L^\infty_{\text{loc}}(\mathbb{R}^N; \mathbb{R})$  and  $a(x)$  is positive-definite for every  $x \in \mathbb{R}^N$ , that is,  $\langle a(x)\xi, \xi \rangle > 0$  for every  $x \in \mathbb{R}^N$ ,  $\xi \in \mathbb{C}^N \setminus \{0\}$ .

Here  $\langle \cdot, \cdot \rangle$  is the usual Hermitian product. Under condition (A1) we define the minimal and maximal realization of  $A$  in  $L^p = L^p(\mathbb{R}^N)$  ( $1 < p < \infty$ ) respectively as

$$\begin{cases} A_{p,\min} u := Au, \\ D(A_{p,\min}) := C_0^\infty(\mathbb{R}^N), \\ A_{p,\max} u := Au, \\ D(A_{p,\max}) := \{u \in L^p \cap W_{\text{loc}}^{2,p}(\mathbb{R}^N); Au \in L^p\}. \end{cases}$$

Our interest is the following properties of  $A_{p,\min}$  and  $A_{p,\max}$ :

- essential  $m$ -accretivity of  $A_{p,\min}$
- $m$ -accretivity of  $A_{p,\max}$
- $m$ -sectoriality of  $A_{p,\max}$

(see e.g., Goldstein [7]). These properties are strongly related to the evolution equation

$$(1.2) \quad \frac{du}{dt} + A_{p,\max} u = 0, \quad t \in (0, \infty), \quad u(0) = u_0.$$

The  $m$ -accretivity of  $A_{p,\max}$  gives solvability of (1.2) and the  $m$ -sectoriality of  $A_{p,\max}$  implies the smoothing effect of solutions to (1.2), which may be expected for equations of parabolic type. Here we only discuss the essential  $m$ -accretivity of  $A_{p,\min}$ .

As is well-known, second-order elliptic operators appear in the theories of non-relativistic quantum mechanics and stochastic analysis. In particular, some important

models of them are written by using the operator  $A$  in (1.1) with unbounded coefficients. For instance, in non-relativistic quantum mechanics the Schrödinger operators

$$Su(x) = -\Delta u(x) + V(x)u(x), \quad x \in \mathbb{R}^N$$

( $a = (\delta_{jk})_{jk}$  and  $F \equiv 0$ , where  $\delta_{jk}$  is the Kronecker delta) describe the motion of a quantum mechanical particle under the potential  $V$ . On the other hand, in stochastic analysis the Ornstein-Uhlenbeck operators

$$A_{OU}u(x) = -\Delta u(x) + Bx \cdot \nabla u(x), \quad x \in \mathbb{R}^N$$

( $a = (\delta_{jk})_{jk}$ ,  $F = (\sum_{j=1}^N B_{jk}x_j)_k$  and  $V \equiv 0$ ) describe the process of random variables, where  $B = (B_{jk})_{jk}$  is an  $N \times N$ -matrix. Our interest is the  $m$ -accretivity of operators which have the differential expression  $A$ .

There exist many investigations dealing with these problems for uniformly elliptic operators with bounded coefficients (see e.g., Kato [9, Example V.3.34], Fattorini [5, Chapter 3], Lunardi [12, Chapter 3] and their references).

For unbounded coefficients, the Schrödinger operators  $-\Delta + V$  have been considered in many previous works (see e.g., Kato [8, 11], Simon [18], Semenov [17], Okazawa [15, 16] and others). The operators  $A$  with unbounded diffusion and drift are also dealt with (see e.g., Cupini-Fornaro [3], Metafuno-Pallara-Prüss-Schnaubelt [13] and Fornaro-Lorenzi [6]).

Here we describe recent results for the (essential)  $m$ -accretivity of  $A_{p,\min}$  and  $A_{p,\max}$  with unbounded diffusions. In Eberle [4], it is shown that under

$$\frac{\langle a(x)x, x \rangle}{|x|^2} \leq \alpha(|x| \log(e + |x|))^2, \quad x \in \mathbb{R}^N \setminus \{0\},$$

the operator  $A_{p,\min}$  with  $F \equiv 0$  and  $V \equiv 0$  is essentially  $m$ -accretive in  $L^p$ . In Metafuno-Pallara-Rabier-Schnaubelt [14], they proved the essential  $m$ -accretivity of  $A_{p,\min}$  under the following conditions: there exist  $\rho \in C^N(\mathbb{R}^N)$  satisfying  $\lim_{|x| \rightarrow \infty} \rho(x) = \infty$  and  $|\nabla \rho| \neq 0$  a.e. on  $\mathbb{R}^N$  and constants  $s, s' > 0$  satisfying  $0 < s' < s$  and

$$(1.3) \quad V - \frac{\operatorname{div} F}{p} \geq 0;$$

$$(1.4) \quad V - \frac{\operatorname{div} F}{p} \geq -s \left( \langle F, \nabla \rho \rangle - \left(1 - \frac{2}{p}\right) \operatorname{div}(a \nabla \rho) \right) + s^2 \langle a \nabla \rho, \nabla \rho \rangle;$$

$$(1.5) \quad e^{-p's'\rho} \langle F, \nabla \rho \rangle \in L^\infty(\mathbb{R}^N);$$

$$(1.6) \quad e^{-p's'\rho} \langle a \nabla \rho, \nabla \rho \rangle \in L^\infty(\mathbb{R}^N).$$

They also deal with the  $m$ -accretivity of  $\tilde{A}_{p,\min}$  under a similar restriction as (1.3)–(1.6) mentioned above. Although their result enables us to deal with general coefficients if  $p = 2$ , a certain restriction on the derivative of the diffusion  $a$  is required when  $p \neq 2$  (see condition (1.4)). Because of this gap between  $p = 2$  and  $p \neq 2$ , it seems to be unnatural from the view point of  $L^p$ -generalization.

In [19], the  $m$ -accretivity of  $A_{p,\max}$  and coincidence of  $A_{p,\max}$  and the closure of  $A_{p,\min}$  are proved if the coefficients  $a$ ,  $F$  and  $V$  satisfy that there exists a nonnegative auxiliary function  $\Psi_p \in L^\infty_{\text{loc}}(\mathbb{R}^N; [0, \infty))$  such that

$$(1.7) \quad \frac{\langle a(x)x, x \rangle}{|x|^2} \leq (1 + \Psi_p(x))^{1-\frac{2}{r}} f(|x|)^2, \quad x \in \mathbb{R}^N \setminus B_R,$$

$$(1.8) \quad \frac{|\langle F(x), x \rangle|}{|x|} \leq (1 + \Psi_p(x))^{1-\frac{1}{r}} f(|x|), \quad x \in \mathbb{R}^N \setminus B_R,$$

$$(1.9) \quad V(x) - \frac{\operatorname{div} F(x)}{p} \geq \Psi_p(x), \quad x \in \mathbb{R}^N$$

with  $f(r) = r \log r$  for constants  $R > 1$  and  $r \in [2, \infty)$ , where  $B_R$  is the  $N$ -dimensional ball with center at the origin and radius  $R$ . More generally, the case where

$$f \in \mathcal{F}_R := \left\{ f \in C([R, \infty); (0, \infty)) ; \int_R^\infty f(s)^{-1} ds = \infty \right\}$$

is dealt with in [20]; note that the case where  $V \equiv 0$  and  $\Psi_p \equiv 0$  is proved in [4, Theorem 2.3] and  $\mathcal{F}_R$  essentially appears in its proof. The optimality of  $\mathcal{F}_R$  is shown in [2, Example 3.5]. This result may be regarded as a natural generalization from  $p = 2$  to  $p \neq 2$ . In [19], the following view point is crucial.

**Proposition 1.1** (see [19, Section 1]). *Assume that (A1) is satisfied. Then for every  $1 \leq q \leq \infty$ ,  $w \in W^{1,1}_{\text{loc}}(\mathbb{R}^N)$  and  $\psi \in C^\infty_0(\mathbb{R}^N)$ ,*

$$(1.10) \quad \begin{aligned} \int_{\mathbb{R}^N} (A\psi)\bar{w} dx &= \int_{\mathbb{R}^N} \left[ \langle a\nabla\psi, \nabla w \rangle + \left( V - \frac{\operatorname{div} F}{q} \right) \psi\bar{w} \right] dx \\ &\quad + \int_{\mathbb{R}^N} \left[ \frac{1}{q'} \langle \bar{w}\nabla\psi, F \rangle - \frac{1}{q} \langle \psi\nabla\bar{w}, F \rangle \right] dx, \end{aligned}$$

where  $q' = \frac{q}{q-1}$  (the Hölder conjugate of  $q$ ).

The equality (1.10) may be regarded as a generalization in  $L^p$  of decomposition formula for sesquilinear form in  $L^2$  into symmetric and anti-symmetric parts.

Recently, in [21], the endpoint case  $r = \infty$  of [19] is discussed under an additional condition similar to the oscillation condition  $|\nabla\Psi_p|^2 \leq \gamma\Psi_p^3$  (see (1.15) below).

On the other hand, in Kato [10], the essential selfadjointness of the Schrödinger operators  $(-\operatorname{div}(a\nabla\cdot) + V)_{2,\min}$  with the following coefficients is posed:

$$(K) \quad \begin{cases} \frac{\langle a(x)x, x \rangle}{|x|^2} \leq k(1 + |x|)^{\ell+2}, & x \in \mathbb{R}^N \setminus B_R, \\ V(x) \geq c|x|^\ell, & x \in \mathbb{R}^N, \end{cases}$$

with  $k, c, \ell > 0$ . This problem is partially solved in [14] under the additional condition  $c > \ell^2/4$ . In the case  $c < \ell^2/4$ , the negative answer (counterexample) is given in [22] which is written in  $L^p$ -framework.

The first purpose of this paper is to prove the assertions of the essential  $m$ -accretivity of  $A_{p,\min}$  in [19] and [21] via a unified approach. The second is to give a summary of answer to Kato's selfadjointness problem and its proof in  $L^2$ -framework (which is simpler than [22]).

Here we introduce the main assumption of this paper (almost the same setting as (1.7)–(1.9)).

**(A2)** There exist constants  $\alpha, \beta > 0, r \in [2, \infty], R > 0$  and a nonnegative auxiliary function  $\Psi_p \in L^\infty_{\text{loc}}(\mathbb{R}^N; \mathbb{R})$  such that

$$(1.11) \quad \frac{\langle a(x)x, x \rangle}{|x|^2} \leq \alpha(1 + \Psi_p(x))^{1-\frac{2}{r}} (|x| \log |x|)^2 \quad \text{a.a. } x \in \mathbb{R}^N \setminus B_R;$$

$$(1.12) \quad \frac{\langle F(x), x \rangle}{|x|} \leq \beta(1 + \Psi_p(x))^{1-\frac{1}{r}} (|x| \log |x|) \quad \text{a.a. } x \in \mathbb{R}^N \setminus B_R;$$

$$(1.13) \quad V - \frac{\operatorname{div} F}{p} \geq \Psi_p \quad \text{a.e. on } \mathbb{R}^N.$$

Now we are in a position to state our main result. The first theorem is the assertion for essential  $m$ -accretivity in the case where  $r \in [2, \infty]$ .

**Theorem 1.1** ([19, Theorem 1.1]). *Let  $1 < p < \infty$ . Assume that (A1) and (A2) are satisfied with  $r \in [2, \infty]$ . Then  $A_{p,\min}$  is essentially  $m$ -accretive in  $L^p$ , that is,*

$$(1.14) \quad \operatorname{Re} \int_{\mathbb{R}^N} (A_{p,\min} u) \bar{u} |u|^{p-2} dx \geq 0 \quad \forall u \in D(A_{p,\min}), \quad \overline{R(1 + A_{p,\min})} = L^p,$$

where  $R(1 + A_{p,\min})$  is the range of  $1 + A_{p,\min}$ .

The second is the assertion for essential  $m$ -accretivity in the endpoint case  $r = \infty$  of Theorem 1.1.

**Theorem 1.2** ([21, Theorem 1.1]). *Let  $1 < p < \infty$ . Assume that (A1) and (A2) are satisfied with  $r = \infty$ . Assume further that  $\Psi_p \in W^{1,\infty}_{\text{loc}}(\mathbb{R}^N)$  and there exists  $\gamma_p > \frac{p-1}{4}$  such that*

$$(1.15) \quad \Psi_p \geq \frac{1}{p'} \frac{\langle F, \nabla \Psi_p \rangle}{1 + \Psi_p} + \gamma_p \frac{\langle a \nabla \Psi_p, \nabla \Psi_p \rangle}{(1 + \Psi_p)^2} \quad \text{a.e. on } \mathbb{R}^N.$$

Then  $A_{p,\min}$  is essentially  $m$ -accretive in  $L^p$ .

The conditions (1.13) and (1.15) in Theorem 1.2 can be replaced with a weaker condition.

**Theorem 1.3.** *Let  $1 < p < \infty$ . Assume that (A1) and (1.11) and (1.12) are satisfied with  $r = \infty$ . Assume further that  $\Phi_p \in W^{1,\infty}_{\text{loc}}(\mathbb{R}^N)$  and there exists  $\gamma_p > \frac{p-1}{4}$  such that*

$$(1.16) \quad V - \frac{\operatorname{div} F}{p} \geq \max \left\{ 0, \frac{1}{p'} \frac{\langle F, \nabla \Psi_p \rangle}{1 + \Psi_p} + \gamma_p \frac{\langle a \nabla \Psi_p, \nabla \Psi_p \rangle}{(1 + \Psi_p)^2} \right\} \quad \text{a.e. on } \mathbb{R}^N.$$

Then  $A_{p,\min}$  is essentially  $m$ -accretive in  $L^p$ .

The last theorem is a summary of the answer to Kato's selfadjointness problem.

**Theorem 1.4** ([21, Theorem 3.1] and [22, Theorem 1.1],  $p = 2$ ). *The following assertions hold:*

- (i) *If  $(a, V)$  satisfies (A1) and (K) with  $c > \frac{k\ell^2}{4}$ , then  $A_{2,\min}$  is essentially selfadjoint.*
- (ii) *If  $N \geq 5$  and  $c_0 < \frac{k_0\ell^2}{4}$ , then there exists a pair  $(a, V)$  such that  $(a, V)$  satisfies (A1) and (K) with  $k = k_0$  and  $c = c_0$  and  $A_{2,\min}$  is not essentially selfadjoint.*

The plan of this paper is as follows. Theorems 1.1, 1.2 are proved in Section 2 via a unified approach. In Section 3, We prove Theorem 1.4 (i) and (ii). The proof of (i) is based on Theorem 1.3 and the other is a simplified version of that in [22].

## 2. Proofs of Theorems 1.1 and 1.3

First we show that  $A_{p,\min}$  is accretive in  $L^p$  (the first part of (1.14)).

*Proof of Theorems 1.1 and 1.3 (accretivity).* Let  $u \in C_0^\infty(\mathbb{R}^N)$ . If  $2 \leq p < \infty$ , then taking the real part of (1.10) with  $q = p$ ,  $w = |u|^{p-2}u$  and  $\psi = u$ , we see from (A1) and  $V - \frac{\operatorname{div} F}{p} \geq 0$  that

$$\begin{aligned} \operatorname{Re} \int_{\mathbb{R}^N} (Au)\bar{u}|u|^{p-2} dx &= (p-1) \int_{\mathbb{R}^N} |u|^{p-4} \langle a \operatorname{Re}(\bar{u} \nabla u), \operatorname{Re}(\bar{u} \nabla u) \rangle dx \\ &\quad + \int_{\mathbb{R}^N} |u|^{p-4} \langle a \operatorname{Im}(\bar{u} \nabla u), \operatorname{Im}(\bar{u} \nabla u) \rangle dx \\ &\quad + \int_{\mathbb{R}^N} \left( V - \frac{\operatorname{div} F}{p} \right) |u|^p dx \geq 0. \end{aligned}$$

If  $1 < p < 2$ , then we use (1.10) with  $q = p$ ,  $w = (|u|^2 + \varepsilon)^{\frac{p-2}{2}}u$  ( $\varepsilon > 0$ ) and  $\psi = u$ . Letting  $\varepsilon \downarrow 0$ , we obtain the accretivity of  $A_{p,\min}$  for  $1 < p < 2$ .  $\square$

Next we prove the (essential) maximality of  $A_{p,\min}$ , that is,  $R(1 + A_{p,\min})$  (the range of  $1 + A_{p,\min}$ ) is dense in  $L^p$ . We only prove the case  $2 \leq p' < \infty$  ( $1 < p \leq 2$ ) in order to avoid the complicated computation. The case  $1 < p' < 2$  can be verified via a procedure similar to the other case with  $\varepsilon$ -regularization as in the proof of accretivity (see [19, Theorem 1.1], [20, Theorem 1.1] and [21, Theorem 1.1]).

Here we need the following lemma.

**Lemma 1.** *Let  $v \in L^{p'}$  be real-valued. Assume that (A1) and*

$$(2.1) \quad \int_{\mathbb{R}^N} v(\varphi + A\varphi) dx = 0 \quad \forall \varphi \in C_0^\infty(\mathbb{R}^N).$$

*Then  $v \in H_{\operatorname{loc}}^1(\mathbb{R}^N) \cap C(\mathbb{R}^N)$ . Moreover, if  $\Phi \in W^{1,\infty}(\mathbb{R}^N)$  has a compact support in  $\mathbb{R}^N$ , then*

$$\begin{aligned} \int_{\mathbb{R}^N} \left[ (p'-1) \langle a \nabla v, \nabla v \rangle \Phi |v|^{p-2} + \langle a \nabla v, \nabla \Phi \rangle v |v|^{p-2} \right] dx \\ + \int_{\mathbb{R}^N} \left[ \frac{1}{p'} \langle F, \nabla \Phi \rangle + \left( 1 + V - \frac{\operatorname{div} F}{p} \right) \Phi \right] |v|^{p'} dx = 0. \end{aligned}$$

*Proof.* Using the elliptic regularity (see e.g., Agmon [1, Lemma 5.1]) iteratively, we see  $v \in H_{\text{loc}}^1(\mathbb{R}^N) \cap C(\mathbb{R}^N)$ . Then using (1.10) with  $q = p$  and  $w = v$  and  $\psi = \varphi$ , we deduce

$$\int_{\mathbb{R}^N} \left[ \langle a \nabla v, \nabla \varphi \rangle + \frac{1}{p'} \langle F, v \nabla \varphi \rangle - \frac{1}{p} \langle F, \varphi \nabla v \rangle + \left( 1 + V - \frac{\text{div} F}{p} \right) v \varphi \right] dx = 0.$$

The above equality is verified even for  $\varphi \in H^1(\mathbb{R}^N)$  with a compact support. Here we choose  $\varphi = \Phi v |v|^{p'-2}$ . Then noting that

$$\frac{1}{p'} v \nabla \varphi - \frac{1}{p} \varphi \nabla v = \frac{1}{p'} \nabla \Phi |v|^{p'},$$

we obtain the desired assertion.  $\square$

*Proof of Theorem 1.1 (maximality).* Assume (2.1) for  $v \in L^{p'}(\mathbb{R}^N)$ . It suffices to prove that  $v = 0$  a.e. on  $\mathbb{R}^N$ . We may assume without loss of generality that  $v$  is real-valued. We take the cut-off functions  $\{\zeta_n\}_n \subset W^{1,\infty}(\mathbb{R}^N)$  as

$$\zeta_n(x) := \begin{cases} 1 & \text{if } |x| < \exp \exp n, \\ 0 & \text{if } |x| > \exp \exp(n+1), \\ n+1 - \log \log |x| & \text{otherwise} \end{cases}$$

for  $n \in \mathbb{N}$  and  $x \in \mathbb{R}^N$ . Applying Lemma 1 with  $\Phi = \zeta_n^r$ , we deduce that

$$\begin{aligned} & (p' - 1) \int_{K_n} \zeta_n^r \langle a \nabla v, \nabla v \rangle |v|^{p'-2} dx + r \int_{K_n \setminus K_{n-1}} \zeta_n^{r-1} \langle a \nabla v, \nabla \zeta_n \rangle |v|^{p'-2} v dx \\ & + \frac{r}{p'} \int_{K_n \setminus K_{n-1}} \zeta_n^{r-1} \langle F, \nabla \zeta_n \rangle |v|^{p'} dx + \int_{K_n} \zeta_n^r \left( 1 + V - \frac{\text{div} F}{p} \right) |v|^{p'} dx = 0, \end{aligned}$$

where  $K_n := \text{supp } \zeta_n$ . By the Cauchy-Schwarz and Young inequalities, we have

$$\begin{aligned} (2.2) \quad \int_{K_n} \zeta_n^r \left( 1 + V - \frac{\text{div} F}{p} \right) |v|^{p'} dx & \leq \frac{r^2}{4(p' - 1)} \int_{K_n \setminus K_{n-1}} \zeta_n^{r-2} \langle a \nabla \zeta_n, \nabla \zeta_n \rangle |v|^{p'} dx \\ & - \frac{r}{p'} \int_{K_n \setminus K_{n-1}} \zeta_n^{r-1} \langle F, \nabla \zeta_n \rangle |v|^{p'} dx. \end{aligned}$$

On the other hand, note that

$$\nabla \zeta_n(x) = \begin{cases} \frac{-x}{|x| \log |x|} & \text{if } x \in K_n \setminus K_{n-1}, \\ 0 & \text{otherwise.} \end{cases}$$

Thus it follows from (1.11), (1.12) of the condition (A2) and Young's inequality that there exist constants  $C_1, C_2 > 0$  such that for every  $n \geq \log \log R$ ,

$$\begin{aligned} \zeta_n^{r-2} \langle a \nabla \zeta_n, \nabla \zeta_n \rangle & = \frac{\zeta_n^{r-2} \langle a(x)x, x \rangle}{|x|^4 (\log |x|)^2} \leq \alpha \zeta_n^{r-2} (1 + \Psi_p)^{1-\frac{2}{r}} \leq \frac{2(p' - 1)}{r^2} (C_1 + \zeta_n^r (1 + \Psi_p)), \\ -\zeta_n^{r-1} \langle F, \nabla \zeta_n \rangle & = \frac{\zeta_n^{r-1} \langle F(x), x \rangle}{|x|^2 \log |x|} \leq \beta (1 + \Psi_p)^{1-\frac{1}{r}} \zeta_n^{r-1} \leq \frac{p'}{2r} (C_2 + \zeta_n^r (1 + \Psi_p)). \end{aligned}$$

Therefore, combining (2.2), (1.13) and the above estimates, we have

$$\int_{K_n} \zeta_n^r (1 + \Psi_p) |v|^{p'} dx \leq (C_1 + C_2) \int_{K_n \setminus K_{n-1}} |v|^{p'} dx + \int_{K_n \setminus K_{n-1}} \zeta_n^r (1 + \Psi_p) |v|^{p'} dx.$$

Consequently, we see that

$$\int_{K_{n-1}} |v|^{p'} dx \leq (C_1 + C_2) \int_{K_n \setminus K_{n-1}} |v|^{p'} dx \rightarrow 0$$

as  $n \rightarrow \infty$ . This implies that  $v = 0$  a.e. on  $\mathbb{R}^N$ , that is,  $R(1 + A_{p,\min})$  is dense in  $L^p$ .  $\square$

*Proof of Theorem 1.3 (maximality).* Assume (2.1) for real-valued function  $v \in L^{p'}$ . As in the proof of Theorem 1.1, we prove that  $v = 0$  a.e. on  $\mathbb{R}^N$ . Applying Lemma 1 with  $\Phi = \Theta_p^{-1} \zeta_n^2$  ( $\Theta_p := 1 + \Psi_p$ ), we deduce that

$$(2.3) \quad (p' - 1) \int_{K_n} \frac{\zeta_n^2 \langle a \nabla v, \nabla v \rangle |v|^{p'-2}}{\Theta_p} dx + 2 \int_{K_n \setminus K_{n-1}} \frac{\zeta_n \langle a \nabla v, \nabla \zeta_n \rangle |v|^{p'-2} v}{\Theta_p} dx \\ - \int_{K_n} \frac{\zeta_n^2 \langle a \nabla v, \nabla \Psi_p \rangle |v|^{p'-2} v}{\Theta_p^2} dx - \frac{1}{p'} \int_{K_n \setminus K_{n-1}} \frac{\zeta_n^2 \langle F, \nabla \Psi_p \rangle |v|^{p'}}{\Theta_p^2} dx \\ + \frac{2}{p'} \int_{K_n \setminus K_{n-1}} \frac{\zeta_n \langle F, \nabla \zeta_n \rangle |v|^{p'}}{\Theta_p} dx + \int_{K_n} \frac{\zeta_n^2}{\Theta_p} \left( 1 + V - \frac{\operatorname{div} F}{p} \right) |v|^{p'} dx = 0.$$

By the Cauchy-Schwarz and Young inequalities, we have

$$\int_{K_n} \frac{\zeta_n^2}{\Theta_p} \left( 1 + V - \frac{\operatorname{div} F}{p} - \frac{1}{p'} \frac{\langle F, \nabla \Psi_p \rangle}{\Theta_p} - \gamma_p \frac{\langle a \nabla \Psi_p, \nabla \Psi_p \rangle}{\Theta_p^2} \right) |v|^{p'} dx \\ = -(p' - 1) \int_{K_n} \frac{\zeta_n^2 \langle a \nabla v, \nabla v \rangle |v|^{p'-2}}{\Theta_p} dx - \gamma_p \int_{K_n} \frac{\zeta_n^2 \langle a \nabla \Psi_p, \nabla \Psi_p \rangle |v|^{p'}}{\Theta_p^3} dx \\ - \int_{K_n} \frac{\zeta_n^2 \langle a \nabla v, \nabla \Psi_p \rangle |v|^{p'-2} v}{\Theta_p^2} dx - 2 \int_{K_n \setminus K_{n-1}} \frac{\zeta_n \langle a \nabla v, \nabla \zeta_n \rangle |v|^{p'-2} v}{\Theta_p} dx \\ - \frac{2}{p'} \int_{K_n \setminus K_{n-1}} \frac{\zeta_n^2 \langle F, \nabla \zeta_n \rangle |v|^{p'}}{\Theta_p} dx \\ \leq - \left( p' - 1 - \frac{1}{4\gamma_p} \right) \int_{K_n} \frac{\zeta_n^2 \langle a \nabla v, \nabla v \rangle |v|^{p'-2}}{\Theta_p} dx \\ - 2 \int_{K_n \setminus K_{n-1}} \frac{\zeta_n \langle a \nabla v, \nabla \zeta_n \rangle |v|^{p'-2} v}{\Theta_p} dx - \frac{2}{p'} \int_{K_n \setminus K_{n-1}} \frac{\zeta_n \langle F, \nabla \zeta_n \rangle |v|^{p'}}{\Theta_p} dx \\ \leq \left( p' - 1 - \frac{1}{4\gamma_p} \right)^{-1} \int_{K_n \setminus K_{n-1}} \frac{\langle a \nabla \zeta_n, \nabla \zeta_n \rangle |v|^{p'}}{\Theta_p} dx - \frac{2}{p'} \int_{K_n \setminus K_{n-1}} \frac{\zeta_n \langle F, \nabla \zeta_n \rangle |v|^{p'}}{\Theta_p} dx.$$

Therefore we see from (1.11), (1.12) and (1.16) that

$$\int_{K_n} \frac{\zeta_n^2}{\Theta_p} |v|^{p'} dx \leq \left[ \alpha \left( p' - 1 - \frac{1}{4\gamma_p} \right)^{-1} + \frac{2\beta}{p'} \right] \int_{K_n \setminus K_{n-1}} |v|^{p'} dx \rightarrow 0$$

as  $n \rightarrow \infty$ . This implies that  $v = 0$  a.e. on  $\mathbb{R}^N$ . This completes the proof.  $\square$



*Remark 2.1.* The difference between the proof of Theorem 1.1 and one of Theorem 1.1 is only the choice of sequence of  $\Phi$ . If  $r \in [2, \infty)$ , then we do not need to assume the differentiability of  $\Psi_p$ . On the other hand, if  $r = \infty$ , then the differentiability of  $\Psi_p$  is required.

### 3. Proof of Theorem 1.4

*Proof of Theorem 1.4 (i).* To apply Theorem 1.2 with  $p = 2$ , we put

$$\Psi_2(x) := \max\{1, c|x|^\ell\} - 1.$$

Then we see from (K) that for every  $x \in \mathbb{R}^N$  satisfying  $|x| > c^{-1/\ell}$ ,

$$\begin{aligned} \frac{\langle a(x)\nabla\Psi_2(x), \nabla\Psi_2(x) \rangle}{(1 + \Psi_2(x))^2} &= \ell^2 \frac{\langle a(x)x, x \rangle |x|^{2\ell-4}}{c^2|x|^{2\ell}} \\ &\leq k\ell^2|x|^\ell \\ &\leq \frac{k\ell^2}{c}\Psi_p(x). \end{aligned}$$

Therefore if  $c > k\ell^2/4$ , then Theorem 1.2 is applicable to  $(a, V)$ , that is,  $A_{2,\min}$  is essentially  $m$ -accretive in  $L^2$ . Since  $A_{2,\min}$  is symmetric, we obtain the essential selfadjointness of  $A_{2,\min}$ .  $\square$

To give a clear proof, we show Theorem 1.4 (ii) for  $\ell \in (0, 2]$  (For the other case, see [22]). Before starting, we give a strategy of the proof. The proof is divided into the following three parts.

**Step 1.** We consider the Schrödinger operators

$$(3.1) \quad B = -\Delta_y + \frac{\lambda}{y_N^2} + W(y) \quad \text{in } \mathbb{R}_+^N := \mathbb{R}^{N-1} \times (0, \infty)$$

with  $0 \leq W \in C(\overline{\mathbb{R}_+^N})$  and prove that  $B_{2,\min}$  ( $B$  defined on  $C_0^\infty(\mathbb{R}_+^N)$ ) is nonnegative but not essentially selfadjoint in  $L^2(\mathbb{R}_+^N)$  when  $\lambda \in (-\frac{1}{4}, \frac{3}{4})$ .

**Step 2.** Using diffeomorphism  $\Phi : \mathbb{R}^N \rightarrow \mathbb{R}^N$ , we translate the operator  $B$  in  $\mathbb{R}_+^N$  into an operator  $A$  in  $\mathbb{R}^N$ ; remark that  $A$  is not essentially selfadjoint in  $L^2(\mathbb{R}^N)$ .

**Step 3.** We construct  $(a, V)$  such that (K) is satisfied with  $k = k_0$  and  $c = c_0$  and corresponding operator  $A_{2,\min}$  is not essentially selfadjoint in  $L^2(\mathbb{R}^N)$ .

#### 3.1. Step 1

**Lemma 2.** Assume  $-\frac{1}{4} < \lambda < \frac{3}{4}$ . Then  $B_{2,\min}$  is nonnegative but not essentially selfadjoint in  $L^2(\mathbb{R}_+^N)$ .

*Proof. (nonnegativity)* Let  $v \in C_0^\infty(\mathbb{R}_+^N)$ . Then by integration by parts and the one-dimensional Hardy inequality (with respect to  $y_N$ ), we have

$$\begin{aligned} \int_{\mathbb{R}_+^N} (B_{2,\min} v) \bar{v} \, dy &= \int_{\mathbb{R}_+^N} |\nabla v|^2 \, dy + \lambda \int_{\mathbb{R}_+^N} \frac{|v|^2}{y_N^2} \, dy \\ &\geq \int_{\mathbb{R}_+^N} \left| \frac{\partial v}{\partial y_N} \right|^2 \, dy + \lambda \int_{\mathbb{R}_+^N} \frac{|v|^2}{y_N^2} \, dy + \int_{\mathbb{R}_+^N} W |v|^2 \, dy \\ &\geq \left( \lambda + \frac{1}{4} \right) \int_{\mathbb{R}_+^N} \frac{|v|^2}{y_N^2} \, dy + \int_{\mathbb{R}_+^N} W |v|^2 \, dy \\ &\geq 0. \end{aligned}$$

Therefore we can find the Friedrichs extension  $B_F$  of  $B_{2,\min}$ . We remark that  $D(B_F) \subset D(1/y_N)$ . More precisely,

$$\left( \lambda + \frac{1}{4} \right) \int_{\mathbb{R}_+^N} \frac{|v|^2}{y_N^2} \, dy \leq \|v\|_{L^2(\mathbb{R}_+^N)} \|B_F v\|_{L^2(\mathbb{R}_+^N)} \quad \forall v \in D(B_F).$$

**(Non-selfadjointness)** First we prove Fix  $\eta \in C_0^\infty(\mathbb{R}; [0, 1])$  such that  $\eta(s) = 1$  for  $|s| \leq 1$  and  $\eta(s) = 0$  for  $|s| \geq 2$  and set

$$\psi(y) = (y_N)^{\frac{1}{2} - \sqrt{\lambda + \frac{1}{4}}} \prod_{j=1}^N \eta(y_j).$$

Then noting that  $\lambda \in (-\frac{1}{4}, \frac{3}{4})$ , we have  $\psi \in L^2(\mathbb{R}_+^N)$  and  $y_N^{-1} \psi \notin L^2(\mathbb{R}_+^N)$  and therefore  $\psi \notin D(B_F)$ . Define

$$f := \psi + B\psi.$$

Then noting that  $(-\frac{d^2}{ds^2} + \frac{\lambda}{s^2}) s^{\frac{1}{2} - \sqrt{\lambda + \frac{1}{4}}} = 0$ , we have

$$|B\psi(y)| \leq C \left( 1 + (y_N)^{\frac{1}{2} - \sqrt{\lambda + \frac{1}{4}}} \right) \chi_{(-2,2)^{N-1} \times (0,1)}$$

and hence  $f \in L^2(\mathbb{R}_+^N)$ . Setting  $\psi_W := \psi - (1 + B_F)^{-1} f \notin D(B_F)$ , we have  $\psi_W + B\psi_W = 0$ . This yields

$$\int_{\mathbb{R}_+^N} (v + B_{2,\min} v) \psi_W \, dx = \int_{\mathbb{R}_+^N} v (\psi_W + B\psi_W) \, dx = 0.$$

Hence  $B_{2,\min}$  is not essentially selfadjoint in  $L^2(\mathbb{R}_+^N)$ . □

### 3.2. Step 2

The following elementary lemma gives a transform of operators in  $\mathbb{R}^N$  into one in  $\mathbb{R}_+^N$ .

**Lemma 3.** Let  $\Phi \in C^\infty(\mathbb{R}_+^N; \mathbb{R}^N)$  be a diffeomorphism with  $\det D\Phi(y) = y_N^{1/2}$ . Set  $J : C_0^\infty(\mathbb{R}^N) \rightarrow C_0^\infty(\mathbb{R}_+^N)$  as

$$(3.2) \quad Ju(y) := y_N^{1/4} u(\Phi(y)).$$

Then  $J$  can be extended to an isometry from  $L^2(\mathbb{R}^N)$  to  $L^2(\mathbb{R}_+^N)$  and for every  $u \in C_0^\infty(\Omega_2)$ ,

$$(3.3) \quad -\operatorname{div}(a^\Phi \nabla u) = J^{-1} \left( -\Delta - \frac{1}{4y_N^2} \right) Ju,$$

where  $a^\Phi \in C^\infty(\Omega_2, \mathbb{R}^{N^2})$  is defined as  $a^\Phi(x) = (D\Phi D\Phi^*)(\Phi^{-1}(x))$ .

### 3.3. Step 3

Now we consider the operator

$$A^\Phi u = -\operatorname{div}(a^\Phi \nabla u) + c|x|^\ell u,$$

where  $\Phi$  is determined later. Then we have

**Lemma 4.** Condition **(K)** for  $a^\Phi$  is equivalent to the following condition:

$$(3.4) \quad \frac{1}{4} |\nabla |\Phi(y)|^2|^2 \leq k(1 + |\Phi(y)|)^{\ell+2} |\Phi(y)|^2 \quad \text{if } |\Phi(y)| \geq R.$$

Now we introduce a suitable  $\Phi$ . We define  $\Phi \in C^\infty(\mathbb{R}_+^N; \mathbb{R}^N)$  as follows: for  $y = (y', y_N)$  with  $y' \in \mathbb{R}^{N-1}$  and  $y_N > 0$ ,

$$(3.5) \quad \Phi_j(y) := \left( \frac{(y_N)^{1/2}}{F'(y_N)} \right)^{\frac{1}{N-1}} y', \quad \Phi_N(y) := F(y_N).$$

Here  $F \in C^\infty(\mathbb{R}_+; \mathbb{R})$  satisfies  $F'(t) > 0$  and

$$(3.6) \quad F(t) = \begin{cases} -\left(\frac{\ell}{2}t\right)^{-\frac{2}{\ell}} & \text{if } 0 < t < \frac{2}{\ell}, \\ t^{3/2} & \text{if } \frac{4}{\ell} < t < \infty. \end{cases}$$

*Remark 3.1.* In view of (3.4), the choice of  $\Phi_N = F$  may be essential because of

$$\frac{1}{4} \left| \frac{d}{dt} |F(t)|^2 \right|^2 = |F(t)|^{\ell+4}, \quad t \in \left( 0, \frac{2}{\rho-2} \right).$$

This property will be used in Lemma 5.

Next we verify **(K)** for  $a^\Phi$  with precise constant  $k > 0$ .

**Lemma 5.** If  $N \geq 5$ , then there exists  $R_0 > 0$  such that  $a^\Phi$  satisfies **(K)** with  $k = 1$ .

*Proof.* Observe that  $N \geq 3 + 2\beta$  yields  $4\tilde{\beta} \leq 1$  for every  $\rho > 2$  and hence  $k_0 = 1$ . Therefore we only prove the general case.

By virtue of Lemma 4, it suffices to prove **(K)** that (3.4) holds with  $k = 1$  and for some  $R_0$ . By the definition of  $\Phi$ , we see that

$$(3.7) \quad \nabla|\Phi(y)|^2 = 2 \begin{pmatrix} \left( \frac{(y_N)^{1/2}}{F'(y_N)} \right)^{\frac{1}{N-1}} \hat{\Phi}(y) \\ \frac{1}{N-1} \left( \frac{1}{2y_N} - \frac{F''(y_N)}{F'(y_N)} \right) |\hat{\Phi}(y)|^2 + F'(y_N) \Phi_N(y) \end{pmatrix},$$

where  $\hat{\Phi}(y) = (\Phi_1(y), \dots, \Phi_{N-1}(y))$ . Here we prove (3.4) by dividing its proof into three cases:

**(The case  $y_N \geq \frac{4}{\ell}$ ).** In this case  $F(y_N) = (y_N)^{\beta+1}$  and hence we have

$$(3.8) \quad \nabla|\Phi(y)|^2 = 2 \begin{pmatrix} \left( \frac{2}{3} \right)^{\frac{1}{N-1}} \hat{\Phi}(y) \\ \frac{3}{2} |\Phi_N(y)|^{\frac{4}{3}} \end{pmatrix},$$

Therefore there exists  $R_1 > 0$  such that (3.4) holds with  $k = 1$  if  $|\Phi(y)| \geq R_1$ .

**(The case  $\frac{2}{\ell} \leq y_N < \frac{4}{\ell}$ ).** In this case,  $F'$ ,  $1/F'$  and  $F''$  are uniformly bounded on  $[\frac{2}{\rho-2}, \frac{4}{\rho-2}]$ . Hence

$$\frac{1}{2} |\nabla|\Phi(y)|^2| \leq C_1 |\Phi(y)| + C_2 |\hat{\Phi}(y)|^2 \leq (C_1 + C_2 |\Phi(y)|) |\Phi(y)|,$$

where

$$C_1 = \max \left\{ \sup_{t \in [\frac{2}{\ell}, \frac{4}{\ell}]} \left( \frac{t^{a/2}}{F'(t)} \right)^{\frac{1}{N-1}}, \sup_{t \in [\frac{2}{\ell}, \frac{4}{\ell}]} F'(t) \right\},$$

$$C_2 = \frac{1}{N-1} \sup_{t \in [\frac{2}{\ell}, \frac{4}{\ell}]} \left| \frac{1}{2t} - \frac{F''(t)}{F'(t)} \right|.$$

Thus there exists  $R_2 > 0$  such that (3.4) holds with  $k = 1$  if  $|\Phi(y)| \geq R_2$ .

**(The case  $0 < y_N < \frac{2}{\ell}$ ).** Observe that

$$F'(t) = |F(t)|^{\frac{\ell+2}{2}}, \quad F''(t) = -\frac{\rho}{2} |F(t)|^{\ell+1}.$$

Hence

$$\nabla|\Phi(y)|^2 = 2 \begin{pmatrix} \left( \frac{\ell}{2} \right)^{-\frac{1}{2(N-1)}} |\Phi_N(y)|^{-\beta} \hat{\Phi}(y) \\ \beta |\Phi_N(y)|^{\frac{\ell}{2}} |\hat{\Phi}(y)|^2 - 2 |\Phi_N(y)|^{2+\frac{\ell}{2}} \end{pmatrix},$$

where  $\beta = \frac{3\ell+4}{N-1}$ . Noting that  $|\Phi_N(y)| \geq 1$ , we have

$$\left(\frac{\ell}{2}\right)^{-\frac{1}{2(N-1)}} |\Phi_N(y)|^{-\beta} |\hat{\Phi}(y)| \leq \left(\frac{\ell}{2}\right)^{-\frac{1}{2(N-1)}} |\Phi(y)|.$$

On the other hand, putting  $|\Phi_N(y)| = |\Phi(y)|\sqrt{s}$  and  $|\Phi_j(y)'| = |\Phi(y)|\sqrt{1-s}$  ( $s \in [0, 1]$ ), we have

$$\frac{\partial}{\partial x_N} |\Phi(y)|^2 = 2|\Phi(y)|^{\frac{\ell}{2}+1} \left( \beta s^{\frac{\ell-2}{4}} (1-s) - s^{\frac{2+\ell}{4}} \right).$$

By the standard computation, we have

$$\left| \beta s^{\frac{\ell-2}{4}} (1-s) - s^{\frac{2+\ell}{4}} \right| \leq 1.$$

Hence we can choose  $R_3 > 0$  such that (3.4) holds with  $k = 1$  if  $|\phi(y)| \geq R_3$ . Consequently, taking  $R_0 = \max\{R_1, R_2, R_3\}$ , we have (3.4) with  $k = k_0$  and  $R = R_0$ .  $\square$

Next we define

$$W_\ell(y) := |\Phi(y)|^\ell, \quad y \in \mathbb{R}_+^N.$$

Then we have

**Lemma 6.** *Let  $\ell > 0$ . Then there exists  $M > 0$  depending only on  $\ell$  such that if  $y \in \mathbb{R}_+^N$  satisfies  $y_N \leq \frac{2}{\ell}$ , then*

$$(3.9) \quad \frac{4}{\ell^2} \frac{1}{(y_N)^2} \leq W_\ell(y) \leq \frac{4}{\ell^2} \frac{1}{(y_N)^2} + M(y_N)^{\frac{3\ell+4}{2(N-1)}} |y'|^2.$$

*Proof.* By the definition of  $\Phi$  and  $F$ , we have

$$W_\ell(y) = \left( \left(\frac{\ell}{2}\right)^{\frac{2(\ell+2)}{\ell(N-1)}} (y_N)^{\frac{1}{N-1} + \frac{2(\ell+2)}{\ell(N-1)}} |y'|^2 + \left(\frac{\ell}{2} y_N\right)^{-\frac{4}{\ell}} \right)^{\frac{\ell}{2}}.$$

Noting that  $\ell \leq 2$ , the triangle inequality yields

$$\begin{aligned} \left(\frac{\ell}{2} y_N\right)^{-2} &\leq W_\ell(y) \leq \left(\frac{\ell}{2} y_N\right)^{-2} + \left(\frac{\ell}{2}\right)^{\frac{\ell+2}{N-1}} (y_N)^{\frac{3\ell+4}{2(N-1)}} |y'|^2 \\ &\leq \frac{4}{\ell^2} \frac{1}{(y_N)^2} + \left(\frac{\ell}{2}\right)^{\frac{\ell+2}{N-1}} (y_N)^{\frac{\ell(\beta+1)+2}{N-1}} |y'|^2. \end{aligned}$$

This completes the proof.  $\square$

*Proof of Theorem 1.4 (ii).* Setting  $\widetilde{W}_\ell \geq 0$  as

$$\widetilde{W}_\ell(y) := W_\ell(y) - \frac{4}{\ell^2} \frac{1}{y_N^2} \in C(\overline{\mathbb{R}_+^N}),$$

we deduce

$$A^\Phi u = J^{-1} B^\Phi J u$$

with

$$B^\Phi = -\Delta + \left( \frac{4c}{\ell^2} - \frac{1}{4} \right) \frac{1}{y_N^2} + c\widetilde{W}_\ell(y).$$

Applying Lemma 2 to  $\lambda = \frac{4c}{\ell^2} - \frac{1}{4} \in (-\frac{1}{4}, \frac{3}{4})$  and  $W = c\widetilde{W}_\ell$ , we see that  $B_{2,\min}^\Phi$  is not essentially selfadjoint in  $L^2(\mathbb{R}_+^N)$ . Since  $J$  is an isometry,  $A_{2,\min}^\Phi$  is not essentially selfadjoint in  $L^2(\mathbb{R}^N)$ . This completes the proof.  $\square$

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